Burnside's Lemma

40-minute lesson

Counting Colorings Up to Symmetry

Burnside's Lemma counts distinct objects when rotations and reflections can make two arrangements equivalent.

THE AVERAGING FORMULA

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} \text{Fix}(g)$$

WHY THE AVERAGE WORKS

Count all pairs (g, x) for which symmetry g fixes configuration x. Counting by symmetry gives the sum of Fix(g). Counting the same pairs orbit by orbit gives |G| for every orbit. Dividing by |G| therefore leaves exactly the number of orbits.

Lesson route

1

List the symmetries

Use the full group: identity, rotations, and reflections when appropriate.

2

Count fixed configurations

For each symmetry, count arrangements that remain exactly unchanged.

3

Average

Add all fixed counts and divide by the number of symmetries.

4

Interpret

The average is the number of symmetry classes, also called orbits.

Concept Reminder

1. What Burnside's Lemma Counts

Let X be the finite set of all labeled configurations, and let G be the finite group of allowed symmetries acting on X. Two configurations belong to the same orbit when an allowed symmetry maps one to the other. The notation X/G means the set of these orbits. For g in G, Fix(g) is the number of configurations x in X for which g(x) = x.

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} \mathrm{Fix}(g)$$

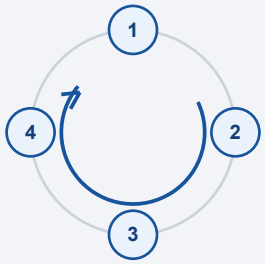
CHECK Here |G| is the number of allowed symmetries. Add the fixed counts first, then divide by |G|.

2. The Cycle Rule

A coloring stays fixed under a symmetry only when every position in the same cycle has the same color. If a symmetry has c cycles and every position may use any of q colors, then it fixes q^c colorings.

$$\mathrm{Fix}(g) = q^c$$

CHECK If the problem requires an exact number of one color, choose only collections of cycles whose lengths add to that number.

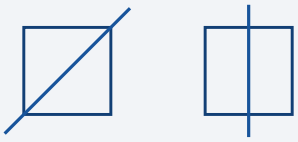


cycle: (1 2 3 4)

3. Choose the Correct Symmetry Group

A necklace problem usually identifies rotations. A bracelet problem identifies rotations and reflections. A square has 8 symmetries: 4 rotations and 4 reflections. For an even-sided figure, reflections may have two different cycle structures.

CHECK Before counting, list every allowed symmetry exactly once.


vertex axis side axis

Guided Examples

Example 1 | Four-Bead Necklace

Four beads are arranged in a circle. Each bead is red or blue. Two colorings are the same if one is a rotation of the other. How many distinct colorings are possible?

WORK SPACE

Example 2 | Three Colors on a Square

The four vertices of a square are colored using any of three colors. Two colorings are the same if a rotation or reflection maps one to the other. How many distinct colorings are possible?

WORK SPACE

Guided Examples

Example 3 | Exactly Three Red Beads

Six beads are arranged in a circle. Exactly three are red and three are blue. Two colorings are the same if one is a rotation of the other. How many distinct colorings are possible?

WORK SPACE

Example 4 | Five-Bead Bracelet

Five beads are arranged in a bracelet. Exactly two are black and three are white. Two bracelets are the same if one can be obtained from the other by a rotation or flip. How many distinct bracelets are possible?

WORK SPACE

Practice Problems

Problem 1 | Three-Color Necklace

Five beads are arranged in a necklace. Each bead may be colored with any of three colors. Two necklaces are the same if one is a rotation of the other. How many distinct necklaces are possible?

WORK SPACE

Practice Problems

Problem 2 | Two Blue Vertices

Exactly two vertices of a square are blue, and the other two are white. Two colorings are the same if a rotation or reflection maps one to the other. How many distinct colorings are possible?

WORK SPACE

Practice Problems

Problem 3 | Six-Bead Bracelet Challenge

Six beads are arranged in a bracelet. Exactly three are black and three are white. Two bracelets are the same if one can be obtained from the other by a rotation or flip. How many distinct bracelets are possible?

WORK SPACE

Practice Problems

Authentic AMC past problem | 2007 AMC 12B #16

16. Each face of a regular tetrahedron is painted either red, white, or blue. Two colorings are considered indistinguishable if two congruent tetrahedra with those colorings can be rotated so that their appearances are identical. How many distinguishable colorings are possible?

(A) 15 **(B)** 18 **(C)** 27 **(D)** 54 **(E)** 81

WORK SPACE

Practice Problems

Authentic AMC past problem | 2020 AMC 12B #20

20 Two different cubes of the same size are to be painted, with the color of each face being chosen independently and at random to be either black or white. What is the probability that after they are painted, the cubes can be rotated to be identical in appearance?

(A) $\frac{9}{64}$ **(B)** $\frac{289}{2048}$ **(C)** $\frac{73}{512}$ **(D)** $\frac{147}{1024}$ **(E)** $\frac{589}{4096}$

WORK SPACE

Final Checklist

Before you count

- [ ] **1. What objects are considered the same?**
- [ ] **2. Did I list every symmetry exactly once?**
- [ ] **3. For each symmetry, which positions must share a color or label?**
- [ ] **4. Did I obey any exact color-count restrictions?**
- [ ] **5. Did I include the identity symmetry?**
- [ ] **6. Did I divide by the total number of symmetries?**

Burnside's Lemma is reliable because it averages fixed configurations over the whole symmetry group. The hard part is almost always Fix(g): identify the cycles created by each symmetry, then enforce the problem's coloring or labeling restrictions.